Efficient Solution Methods for Inverse Problems with Application to Tomography
Radon Transform and Friends

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Content

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X-Ray Tomography

Assumptions:

- X-Rays travel on straight lines
- Attenuation proportional to path length
- Attenuation proportional to number of photons

Radon Transform

$$\begin{aligned} \triangle I &= -I \triangle t\, f \\ g_L &= \ln(I_0/I_L) = \int_L f\, dt \end{aligned}$$

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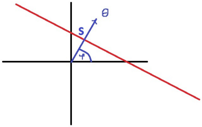
Radon Transform



$$\mathbf{R}f(\theta, s) = \int_{\mathbb{R}^2} f(x)\delta(s - x^\top\theta)dx = \int_{\mathbb{R}} f(s\theta + t\theta^\perp)dt$$

$\theta \in S^1$ and $s \in \mathbb{R}$.
Transform for fixed direction

$$\mathbf{R}_\theta f(s) = \mathbf{R}f(\theta, s)$$

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Radon Transform in Hihger Dimensions

$$\mathbf{R}f(\theta, s) = \int_{\mathbb{R}^N} f(x)\delta(s - x^\top \theta)dx$$

Properties of the Radon Transform

Assume all integrals exist

$$\int_{\mathbb{R}} Rf(\theta, s)\psi(s)ds = \int_{\mathbb{R}^N} f(x)\psi(x^\top\theta)dx$$

Proof:

$$\begin{aligned} \int_{\mathbb{R}} Rf(\theta, s)\psi(s)ds &= \int_{\mathbb{R}}\int_{\mathbb{R}^N} f(x)\delta(s - x^\top\theta)dx\psi(s)ds \\ &= \int_{\mathbb{R}^N} f(x)\int_{\mathbb{R}}\psi(s)\delta(s - x^\top\theta)dsdx \end{aligned}$$

Example: Put

$$\psi(s) = (2\pi)^{-1/2}\exp(-\imath s\sigma)$$

Application

Put $\psi(s) = s^m$, then

$$\int_{\mathbb{R}} \mathbf{R}f(\theta, s)s^m ds = \int_{\mathbb{R}^N} f(x)\left(x^\top\theta\right)^m dx$$
$$= p_m(\theta)$$

where p_m is a polynomial of degree $\leq m$ in θ.
See Consistency Conditions.

Projection Theorem

Fourier Transform

$$\hat{f}(\xi) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} f(x) \exp(-\imath x^\top \xi) dx$$

Projection Theorem

$$\widehat{Rf}(\theta, \sigma) = (2\pi)^{(N-1)/2} \hat{f}(\sigma\theta)$$

Basis for inversion formulas, continuity results etc.

Exercise

Compute the 2D Radon transform of

- Example 1:

$$f(x) = \begin{cases} 1 & : \quad \|x\| \leq 1 \\ 0 & : \quad \text{otherwise} \end{cases}$$

- Example 2:

$$f(x) = \begin{cases} 1 & : \quad \|x\| \leq 1 \text{and} \|x - 0.5\| \geq 0.25 \\ 0 & : \quad \text{otherwise} \end{cases}$$

- Example 3:

$$f(x) = \begin{cases} 1 & : \quad (\frac{x_1}{a})^2 + (\frac{x_2}{b})^2 \leq 1 \\ 0 & : \quad \text{otherwise} \end{cases}$$

Invariance Properties

- Translation: Let $T_1^a f(x) = f(x-a)$. Then

$$RT_1^a = T_2^{a^\top \theta} R$$

  with $T_2^t g(\theta, s) = g(\theta, s-t)$.
- Rotation: Let $T_1 U f(x) = f(Ux)$ where U is a unitary matrix. Then

$$RT_1^U = T_2^U R$$

  where $T_2^U g(\theta, s) = g(U\theta, s)$.

Invariance Properties, Cont'd

- Dilation: $r > 0$

$$\mathbf{R}f(r\theta, rs) = r^{-1}\mathbf{R}f(\theta, s)$$

- General Transform: Let $T_1^A f(x) = f(Ax)$ where A is a regular matrix. Then

$$\mathbf{R}T_1^A = T_2^A\mathbf{R}$$

where

$$T_2^A g(\theta, s) = |\det A| \|A^{-T}\theta\| g\left(\frac{A^{-T}\theta}{\|A^{-T}\theta\|}, \frac{s}{\|A^{-T}\theta\|}\right).$$

Radon Transform and Derivatives

$$\begin{aligned} \mathbf{R} D^\alpha f(\theta, s) &= \theta^\alpha \frac{\partial^{|\alpha|}}{\partial s^{|\alpha|}} \mathbf{R} f(\theta, s) \\ \mathbf{R}\Delta &= \frac{\partial^2}{\partial s^2}\mathbf{R} \end{aligned}$$

Proof: Use Projection Theorem

$$\widehat{\mathbf{R}_\theta D^\alpha f}(\sigma) = (2\pi)^{(N-1)/2}\widehat{(D^\alpha f)}(\sigma\theta)$$

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Radon Transform and Derivatives

$$\mathbf{R}D^{\alpha}f(\theta,s) = \theta^{\alpha}\frac{\partial^{|\alpha|}}{\partial s^{|\alpha|}}\mathbf{R}f(\theta,s)$$

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Proof: Use Projection Theorem

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\widehat{\mathbf{R}_\theta D^{\alpha}f}(\sigma) &= (2\pi)^{(N-1)/2}\widehat{(D^{\alpha}f)}(\sigma\theta) \\
&= (2\pi)^{(N-1)/2}\imath^{|\alpha|}(\sigma\theta)^{\alpha}\hat{f}(\sigma\theta) \\
&= \imath^{|\alpha|}\sigma^{|\alpha|}\theta^{\alpha}\widehat{\mathbf{R}_\theta f}(\sigma) \\
&= \theta^{\alpha}\widehat{D^{|\alpha|}\mathbf{R}_\theta f}(\sigma)
\end{aligned}$$

Continuity

Let $\Omega \subset \mathbb{R}^N$ be bounded, $C^{N-1} = S^{N-1} \times \mathbb{R}$

$$\mathbf{R} : L_2(\Omega) \to L_2(C^{N-1})$$

compact

$$\mathbf{R} : L_2(\Omega) \to H^{(N-1)/2}(C^{N-1})$$

bounded, Natterer, 1979

Content

Ray Transforms

Transverse Ray Transform

$$(\mathcal{P}^\perp w)(\xi, s) = \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \langle w(s\xi + t\xi^\perp), \xi\rangle dt.$$

Longitudinal Ray Transform

$$(\mathcal{P} w)(\xi, s) = \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \langle w(s\xi + t\xi^\perp), \xi^\perp\rangle dt.$$

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For fixed direction ξ the transforms can be rewritten as

$$\begin{aligned} (\mathcal{P}^\perp w)(\xi, s) &= \mathbf{R}_\xi (w^\top \xi)(s) , \\ (\mathcal{P} w)(\xi, s) &= \mathbf{R}_\xi (w^\top \xi^\perp)(s) . \end{aligned}$$

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Questions: Ask Derevtsov, Sharafutdinov !

Decompositions

w can be decomposed uniquely in a sum of potential, solenoidal and harmonic parts (Helmholtz-Hodge decomposition),

$$w = \nabla\varphi + w^s + \nabla h, \quad \operatorname{div} w^s = 0, \quad \varphi|_{\partial B} = 0, \quad \langle w^s, \nu\rangle|_{\partial B} = 0,$$

where $\nabla\varphi$ is a potential vector field, $\varphi \in H_0^1(B)$, w^s is a solenoidal vector field, ∇h is a harmonic vector field, with h to be harmonic function, and ν is the vector of outer to ∂B normal.

Null Spaces

Lemma

For potentials $\varphi \in H_0^1(B)$ the following relations hold

$$\mathcal{P}^{\perp}(\nabla\varphi) = \mathcal{P}(\nabla^{\perp}\varphi) = \frac{\partial}{\partial s}(\mathbf{R}\varphi)$$

and

$$\mathcal{P}(\nabla\varphi) = \mathcal{P}^{\perp}(\nabla^{\perp}\varphi) = 0\ .$$

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$$\mathcal{P}(\nabla\varphi)(\xi, s) = \mathbf{R}_\xi(\nabla\varphi^\top\xi^\perp)(s) = \xi^\top\xi^\perp\frac{\partial}{\partial s}(\mathbf{R}_\xi\varphi)(s) = 0$$

Content

Inversion Formula

Let $\alpha < N$. Then

$$\mathbf{R}^{-1} = \frac{1}{2}(2\pi)^{1-N}\mathbf{I}^{-\alpha}\mathbf{R}^{*}\mathbf{I}^{\alpha+1-N}$$

where $\mathbf{R}^{*}$ is the adjoint operator from L_2 to L_2 known as backprojection

$$\mathbf{R}^{*}g(x) = \int_{S^{N-1}} g(\theta, x^{\top}\theta)d\theta$$

and the Riesz potential $\mathbf{I}^{-\alpha}$ is defined with the Fourier transform

$$\widehat{\mathbf{I}^{-\alpha}g}(\theta,\sigma) = |\sigma|^{\alpha}\hat{g}(\theta,\sigma)$$

Proof

$$\mathbf{I}^\alpha f(x) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{\imath x^\top \xi} |\xi|^{-\alpha} \hat{f}(\xi) d\xi$$

Proof

$$\begin{aligned} \mathbf{I}^\alpha f(x) &= (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{\imath x^\top \xi} |\xi|^{-\alpha} \hat{f}(\xi) d\xi \\ &= (2\pi)^{-N/2} \int_{S^{N-1}} \int_0^\infty e^{\imath \sigma x^\top \theta} \sigma^{N-1-\alpha} \hat{f}(\sigma\theta) d\sigma d\theta \end{aligned}$$

Proof

$$\begin{aligned}
\mathbf{I}^\alpha f(x) &= (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{\imath x^\top \xi} |\xi|^{-\alpha} \hat{f}(\xi) d\xi \\
&= (2\pi)^{-N/2} \int_{S^{N-1}} \int_0^\infty e^{\imath \sigma x^\top \theta} \sigma^{N-1-\alpha} \hat{f}(\sigma\theta) d\sigma d\theta \\
&= (2\pi)^{-N+1/2} \int_{S^{N-1}} \int_0^\infty e^{\imath \sigma x^\top \theta} \sigma^{N-1-\alpha} \widehat{\mathbf{R}f}(\theta, \sigma) d\sigma d\theta
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&= \frac{1}{2} (2\pi)^{-N+1} \int_{S^{N-1}} \mathbf{I}^{\alpha+1-N} \mathbf{R}f(\theta, x^\top \theta) d\theta
\end{aligned}
$$

Local vs. Nonlocal Formulas

N even

$$\mathbf{I}^{1-N} g = (-1)^{(N-2)/2} \mathbf{H} g^{N-1}$$

N odd

$$\mathbf{I}^{1-N} g = (-1)^{(N-1)/2} g^{N-1}$$

where **H** Hilbert transform

$$\mathbf{H} g(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(t)}{s-t} dt$$

Content

Singular Value Decomposition

Consider the Radon transform for $w(s) = (1 - s^2)^{1/2}$

$$R : L_2(\Omega) \to L_2(C^1, w^{-1})$$

Then

$$\sigma_{m\ell} = 2\sqrt{\frac{\pi}{m+1}},\ \ell = 0, \cdots, m, m \geq 0$$

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Problem midly ill-posed

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Singular functions: Zernike and Chebychev polynomials and spherical harmonics

Idea of Derivation

$U(\alpha)$ rotation, $\theta(\varphi) \in S^1$, then

$$U(\alpha)\theta(\varphi) = \theta(\varphi - \alpha)$$

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$$\begin{aligned} \mathbf{R}(r\frac{\partial}{\partial r}f) &= (s\frac{\partial}{\partial s} - 1)\mathbf{R}f \\ \mathbf{R}(r^2\frac{\partial^2}{\partial r^2}f) &= (s^2\frac{\partial^2}{\partial s^2} - 2s\frac{\partial}{\partial s} + 2)\mathbf{R}f \end{aligned}$$

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Spherical Harmonics are eigenfunctions of Laplace-Beltrami operator, then

$$\mathbf{R}D_r(m) = D_s(m)\mathbf{R}$$

Eigenfunctions of the differential operators give radial part.

Limited Angle Transform

Data missing for $|\varphi| \leq \Phi < \pi/2$

$$\sigma_{m\ell} = 2\sqrt{\frac{\pi}{m+1}}\lambda(m, \ell, \Phi), \ \ell = 0, \cdots, m, m \geq 0$$

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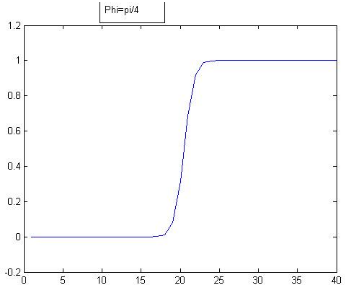

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See SVD or Wavefront Sets (Quinto)

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But some parts can be well recovered
See SVD or Wavefront Sets (Quinto)
Limited Angle Transform $\mathbf{R}_\Phi : L_2(\Omega) \to L_2(S_\Phi \times [-1, 1])$ where
$S_\Phi = \{\theta(\varphi) : |\varphi| \le \Phi\}$ is not rotaional invariant, hence reconstuction kernel
$\psi_x(\theta, s)$ even for mollifier $E(\|x - y\|)$.

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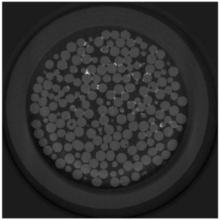
Limited Angle Reconstructionn

0 in missing range:



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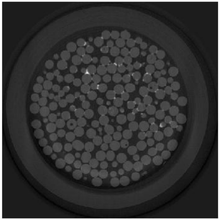

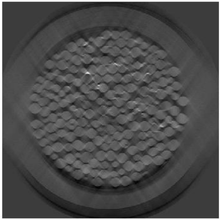

Reconstruction with full and limited data from Syncrotron Measurements

Resolution

- Sampling theorems, Shannon, Petersen-Middleton
- Null Space for finitely many data

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CT: 1.000.000 data

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CT: 1.000.000 data
Resolution: Object size / 1000

Content

Filter

Replace $|\sigma|$ by $|\sigma|F_\gamma(\sigma)$
Shepp-Logan: for $\sigma \leq 1$:

$$F(\sigma/\gamma) = \mathrm{sinc}(\sigma\pi/2)$$

Filter

$$\psi_\gamma(s) = \frac{\gamma}{2\pi^3} u(\gamma s)$$

with

$$u(s) = \frac{\pi/2 - s \sin s}{\pi^2/4 - s^2}$$

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For $s = \ell h$ and $\gamma = \pi/h$, h detector spacing

$$\psi_\gamma(s_\ell) = \frac{\gamma^2}{\pi^4} \frac{1}{1 - 4\ell^2}$$

Algorithm

Based on Approximate Inverse and Trapezoidal Rule

Algorithm

Based on Approximate Inverse and Trapezoidal Rule
Filtered Backprojection

Algorithm

Based on Approximate Inverse and Trapezoidal Rule

Filtered Backprojection

Filtering the data

$$
\begin{aligned}
q(\omega, s) &= \int Rf(\omega, t)\psi_\gamma(s-t)dt \\
&= h\sum_k \psi_\gamma(s-s_k)Rf(\omega_\ell, s_k)
\end{aligned}
$$

Algorithm Continued

Backprojection

$$\begin{aligned} f_\gamma(x) &= \int_{S^1} q(\omega, x^\top \omega)\, d\omega \\ &= \frac{\pi}{L} \sum_{\ell=1}^{L} (1-\eta) q(\omega_\ell, s_k) + \eta q(\omega_\ell, s_{k+1}) \\ kh &\leq x^\top \omega_\ell < (k+1)h \\ \eta &= x^\top \omega_\ell / h - k \end{aligned}$$

Iterative Algorithms

ART and variants

Iterative Algorithms

ART and variants
Preferable for incomplete data
Laminography etc

Iterative Algorithms

ART and variants
Preferable for incomplete data
Laminography etc
EM algorithms for very noisy data (Emission Tomography: SPECT)